To satisfy Eq. (16) exactly for all x in the interval $+1$ to -1, the $B_1 - B_N$ would have to be chosen so as to make the coefficients of each of the polynomials $T_2(x)$ to $T_{4N-2}(x)$ zero. Also, from Eq. (6), the $B_1 - B_N$ must satisfy

$$\sum_{j=1}^{N} (2j-1)B_j = 0 \tag{20}$$

Hence, there are $2N$ equations to be satisfied in all, and for finite N all of these cannot be satisfied exactly by a nontrivial choice of the $B_1 - B_N$. If the conditions that the coefficients of $T_{2N}(x)$ to $T_{4N-2}(x)$ should be zero are abandoned, then the $B_1 - B_N$ may be computed so as to satisfy Eq. (20), together with the $N-1$ equations obtained by comparing the coefficients of $T_2(x)$ to $T_{2N-2}(x)$. These are

$$\sum_{k=1}^{j} (2k-1)B_k = \tfrac{1}{2} \sum_{k=1}^{j} B_k B_{j-k+1} - \sum_{k=1}^{N-j} B_k B_{j+k}$$

$$j=1, =, \ldots, N-1 \tag{21}$$

In view of the mini-max properties of the Chebsyshev polynomials [see Synder[6]] this choice of the $B_1 - B_N$ will satisfy Eq. (16) with an error whose magnitude is the minimum possible on the interval $-1 \leq x \leq +1$.

Equations (20) and (21) have been solved numerically for values of N from 2 to 20. When the magnitude of the maximum error in Eq. (16) was computed, it was found to be less than 1% for $N > 9$. The equation of the bubble surface for the case $N = 9$ was found to be

$$\begin{aligned} y = {} & 0.628t(1-x^2)^{1/2}(0.8242U_0 - 0.4661U_2 \\ & + 0.1817U_4 - 0.0728U_6 + 0.0290U_8 - 0.0116U_{10} \\ & + 0.0046U_{12} - 0.0018U_{14} + 0.0005U_{16}) \end{aligned} \tag{22}$$

and the corresponding results obtained for the vorticity and the constant h were

$$\omega = 5.048t^{-1/2} + 0(t^{1/2}) \tag{23}$$

and

$$h = 1 - 1.256t + 0(t^2) \tag{24}$$

Finally, the sum in the remainder term Eq. (11) was found to be 11.60 for $N = 9$, confirming that this term represents an error of $0(t^2)$.

It should be mentioned that Eqs. (22-24) do not represent the only solution of Eq. (21) for $N = 9$. However, all other solutions give a much larger value of the maximum error in Eq. (16) and also do not give a bubble shape which is of the smooth form depicted in Fig. 1.

For computing the bubble shape it is convenient to express Eq. (22) as a polynomial in x^2. The result is

$$\begin{aligned} y = {} & t(1-x^2)^{3/2}(1 - 4.22x^2 + 16.27x^4 - 47.67x^6 \\ & + 93.61x^8 - 112.34x^{10} + 73.53x^{12} - 19.98x^{14}) \end{aligned} \tag{25}$$

The stream function for the flow in the wake may be obtained from Eq. (10) and that for the exterior flow is found by substituting Eq. (1) into Eq. (4) and taking the imaginary part of w. The streamlines have been computed for the case $t = 0.2$ and the resulting flow pattern is shown in Fig. 2.

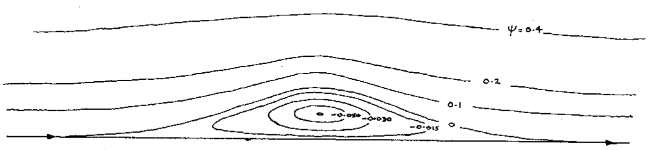

Fig. 2 Streamlines for $t = 0.2$.

Discussion

In the analysis of the preceding sections the shape of a two-dimensional slender bubble has been found [Eq. (25)] which is such that the flow exterior to the bubble is irrotational and the flow in the interior is inviscid with constant vorticity. The bubble constitutes a possible wake bubble for a shell having the same shape as the front portion of the bubble. The extent of the shell (AB Fig. 1) as a fraction of the remaining length of the bubble surface (i.e., the ratio of the length of boundary over which the standing eddy in the wake is retarded to that over which it is driven) should be such as to produce a rate of rotation which is measured by the same value of the vorticity as that given by Eq. (23). To determine this ratio would require a knowledge of the flow in the boundary layers contiguous to the bubble surface. This would involve an extensive analysis and the calculation is, therefore, not attempted here.

Conclusions

A theory is presented for the steady inviscid two-dimensional flow past a sharp nosed body. The body considered is a shell forming the front part of a slender wake bubble, the shape of which is characterized by a cusp at either end. A shape of bubble is computed which is such that the irrotational flow exterior to the bubble and the constant vorticity flow inside the bubble give continuity in pressure across the bubble boundary. The velocity increment across the bubble boundary and the strength of the vorticity are important parameters of the flow and are found to be $1 - 1.256t + 0(t^2)$ and $5.048t^{-1/2} + 0(t^{1/2})$, respectively, where t is the thickness ratio for the bubble.

References

[1] Helmholtz, H.V., "Uber Discontinuierliche Flussigkeitsbewegungen," *Monatsberichte Berlin Akademie,* April 1868, pp. 215-218; reprinted in Philosophical Magazine, Vol. 36, 1868, pp. 337-346.

[2] Kirchoff, G., "Zur Theorie Freier Flussigkeitsstrahlen," *Journal fur de Reine und Angewandte Mathematik,* Vol. 70, heft4, 1869, pp. 289-298.

[3] Batchelor, G.K., "A Proposal Concerning Laminar Wakes behind Bluff Bodies at Large Reynolds Number," *Journal of Fluid Mechanics,* Vol. 1, Oct. 1957, pp. 388-398.

[4] Batchelor, G.K., "On Steady Laminiar Flow with Closed Streamlines at Large Reynolds Number," *Journal of Fluid Mechanics,* Vol. 1, July 1956, pp. 177-190.

[5] Batchelor, G.K., *An Introduction to Fluid Mechanics,* Cambridge University Press, Cambridge, England, 1970, p. 536.

[6] Synder, M.A., *Chebyshev Methods in Numerical Approximation,* Prentice-Hall, Englewood Cliffs, N.J., 1966, p. 54.

Drag Coefficient Equations for Small Particles in High Speed Flows

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Nomenclature

D = diameter, microns
C_D = drag coefficient

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M = Mach number
Re = Reynolds number
t = time, sec
V = velocity, M/sec
x = distance, m
μ_g = gas viscosity, N-sec/M^2
ρ = density, kg/m^3

Subscripts

avg = average particle velocity
g = gas property
max = maximum particle velocity calculated
min = minimum particle velocity calculated
p = particle property

Introduction

THE laser velocimeter (LV) is used to measure flow velocities by measuring the velocities of particles embedded in the flow. Problems arise in the application of the LV system to supersonic flows and to turbulent flows where large velocity gradients may occur. These large velocity gradients can cause significant particle velocity lags, where the particles cannot accelerate or decelerate as rapidly as the gas. The particle motion through regions of large velocity gradients must be examined therefore to determine the magnitude of the particle velocity lag and thus the accuracy of the LV system for velocity measurements in these high speed flows. Particle motion in two-phase flows also depends upon the gas properties, particle properties, and the particle drag coefficient. A literature survey in the area of two-phase flows reveals that a number of drag coefficient equations[1-6] have been used to calculate particle motion in supersonic flows.

The objective of the present investigation is to determine the influence of the various available drag coefficient equations on particle velocity calculations for typical two-phase flows encountered in supersonic LV applications and to compare the predictions of the available drag coefficient equations to experimental sphere drag data. Of particular importance for LV applications is the relative Mach number less than 2 and relative Reynolds number less than 200 regime.

Governing Equations

The one-dimensional equation of motion for a spherical particle traveling in a gas and for the particle density much greater than the gas density is given in Ref. 7 as

$$dV_p/dt = \tfrac{3}{4} C_D Re_p (\mu_g/\rho_p D^2{}_p)\ (V_g - V_p) \tag{1}$$

or

$$d^2x/dt^2 = \tfrac{3}{4} C_D Re_p (\mu_g/\rho_p D^2{}_p)\ (V_g - V_p) \tag{2}$$

If the particle properties, the drag coefficient C_D and the gas properties are known, Eqs. (1) and (2) completely describe the one-dimensional motion of a particle in a two-phase flow. The equations governing particle motion in multi-dimensional flows are given in Ref. 7.

Drag Coefficient Equations

With at least 6 methods available in the literature for determining particle drag coefficients, it is important (particularly in supersonic flow) to determine the sensitivity of the particle motion to the C_D expression employed. Korkan et al.[6] have presented a limited study of the influence of the drag coefficient equations of Ref. 1-3 for a Mach 5 flow where particles: 1) are injected at an angle to the flow, 2) pass through a Prandtl-Meyer expansion fan, or 3) pass through an oblique shock wave. In the present study the C_D equations of Refs. 1-6 are used to determine the influence of the various drag coefficients on particle velocity calculations. Calculations were made for 0.5-10μ particles passing along the centerline of a nozzle, through 5° and 10° oblique shocks in Mach 3-6 flows, or for particles passing through normal shocks in Mach 1.6-6 flows.

Calculations for particle velocity were made using each of the 6 expressions for the particle drag coefficient C_D. To indicate the influence of the drag coefficient on calculated particle velocity, a percent deviation was defined as

$$\%\ \text{deviation} = \frac{V_p,\text{max} - V_p,\text{min}}{V_p,\ \text{avg}} \times 100, \tag{3}$$

The results from these calculations indicated that the equation for the drag coefficient has a large influence on particle velocity for particles passing through normal shocks and for the vertical or y-component of particle velocity behind oblique shocks. For example, the maximum percent deviation at a location 1.27 cm behind the normal shock ranged from 8% for Mach 1.6 flows to 80% for Mach 6 flows. For the calculations of the vertical component of the particle velocity behind oblique shocks, the precent deviation at a location 1.27 cm behind the shock ranged from 26% for Mach 3 flows to 92% for Mach 6 flows. The calculations for center-line nozzle flows as well as the horizontal components of particle velocities behind oblique shocks deviated less than 6%.

The deviation in the calculation of the components of particle velocity is also an indication of the magnitude of the deviation in the calculated trajectories. Obviously, there is no deviation in the particle trajectories for the one-dimensional flow behind a normal shock and long the centerline of a nozzle. However, the velocity deviations in a two-dimensional flow generally cause accompanying deviations in the particle trajectory. For example, at a position 21.32 cm downstream of the oblique shocks mentioned in the previous paragraph the y-position variations are as large as .11 cm at Mach 3 and 1.10 cm at Mach 6.

With large percent deviations or uncertainties possible in the calculated particle motions for representative flows encountered in LV applications, it is necessary to examine the 6 methods for calculating C_D to determine which, if any, of the equations best predicts experimental sphere drag data for the low particle Mach number and low Reynolds number case. The most extensive sphere drag data available are those of Bailey and Hiatt.[8] In the low particle Mach number and Reynolds number range of interest in LV applications, Bailey and Hiatt[8] have obtained sphere drag data for particle Mach numbers 0.12 to 2.0 at a particle Reynolds number of 200, and particle Mach numbers 1.0 to 2.0 for particle Reynolds numbers of 20-200. Zarin[9] has also made subsonic sphere drag measurements for particle Reynolds numbers less than 100. These data from Refs. 8 and 9 then comprise the experimental data base used herein for evaluating the expressions for the particle drag coefficient. In selecting the data of Bailey and Hiatt[8] and Zarin[9] earlier less accurate data have been neglected; the sources of error for these data are support interference, turbulence effects, and other errors peculiar to experimental measurements of particle drag coefficients.[10]

From the present comparisons of the drag coefficient predictions of Refs. 1-6 with the experimental data, the method of Korkan et al.[6] was found to give the best overall agreement. Therefore, this comparison is shown in Fig. 1; however, the differences between predictions and the experimental data are still as large as 20% for even this best method. (Apparently Korkan et al.[6] only used the high Re_p data of Bailey and Hiatt[8]).

Improved Drag Coefficient Equation

The equation used by Cuddihy et al.[1] and later by Korkan et al.[6] is suitable for fitting available experimental drag coefficient data; this equation is

$$C_D = C_{D,C} + (C_{D,FM} - C_{D,C})\, e^{-A Re_p^N} \tag{4}$$

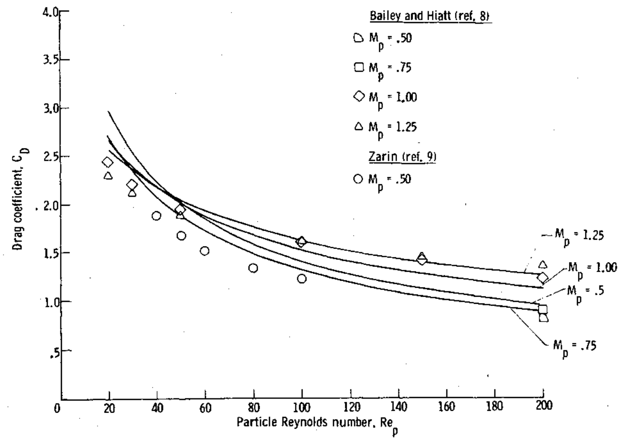

Fig. 1 Comparison of Korkan et al. method[6] with experimental data.

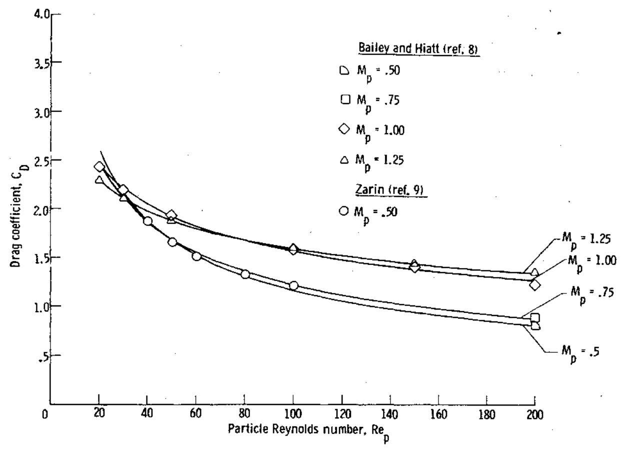

Fig. 2 Comparison of present drag coefficient method with experimental data.

Table 1 Parameters in present drag coefficient expression

M_p	$C_{D,C}$	$C_{D,FM}$	A	N
0.1	0.380	53.541	1.7269	0.1976
0.15	0.381	35.759	1.4099	0.2196
0.2	0.390	26.888	1.1908	0.2399
0.25	0.392	21.580	1.0339	0.2562
0.3	0.398	18.053	0.9144	0.2706
0.35	0.403	15.544	0.8159	0.2846
0.4	0.410	13.670	0.7356	0.2973
0.45	0.419	12.222	0.6672	0.3097
0.5	0.426	11.065	0.6085	0.3215
0.55	0.435	10.125	0.5637	0.3301
0.6	0.443	9.345	0.5244	0.3384
0.65	0.453	8.688	0.4890	0.3467
0.7	0.466	8.128	0.4602	0.3536
0.75	0.480	7.645	0.4367	0.3585
0.8	0.500	7.224	0.4163	0.3630
0.85	0.513	6.853	0.4043	0.3620
0.9	0.540	6.525	0.3909	0.3631
0.95	0.600	6.233	0.7435	0.3096
1.0	0.710	5.970	0.4384	0.3086
1.1	0.780	5.517	0.4332	0.3059
1.2	0.820	5.141	0.4261	0.3036
1.3	0.860	4.823	0.4252	0.3003
1.4	0.890	4.551	0.4260	0.2969
1.5	0.910	4.316	0.4334	0.2895
1.6	0.920	4.110	0.4392	0.2826
1.7	0.930	3.930	0.4483	0.2747
1.8	0.940	3.771	0.4535	0.2696
1.9	0.940	3.630	0.4545	0.2649
2.0	0.940	3.505	0.4489	0.2640

where $C_{D,C}$ and $C_{D,FM}$ are the continuum and free molecular values of the drag coefficient respectively. The parameters A and N in Eq. (4) are evaluated to give the best fit to appropriate experimental data in the particle Mach number and Reynolds number range of interest.

Equation (4) is used herein to fit the experimental data of Bailey and Hiatt[8] and Zarin[9] for particle Mach numbers of 0.1-2. The values for the continuum drag coefficients are also taken from the Bailey and Hiatt[8] high Reynolds number data. The free molecular drag coefficient values are determined from the free molecular drag coefficient equation given by Emmons[11] assuming diffuse reflection. The resultant values for the parameters $C_{D,C}$, $C_{D,FM}$, A, and N are listed in Table 1. Figure 2 shows that Eq. (4) with the new parameters given in Table 1 provides an excellent fit of the experimental data.

The present method is formulated to approach the incompressible values of the drag coefficient as the particle Mach number approaches 0.1. For particle Mach numbers less than 0.1, the Mach number or compressibility effect is thus negligible; an incompressible drag coefficient equation given by Torobin and Gauvin[12] gives good predictions of the incompressible experimental data for $Re_p \leq 200$ and is given by

$$C_D = (24/Re_p)\ (1 + .15\, Re_p^{\ .687}) \tag{5}$$

This new equation should significantly improve the accuracy of particle lag calculations for laser velocimetry in high speed flows.

References

[1] Cuddihy, W. F., Beckwith, I. E., and Schroeder, L. C., "A Solution to the Problem of Communication Blackout of Hypersonic Reentry Vehicles," presented at the Anti-Missile Research Advisory Council Meeting, Annapolis, Md., Oct. 22-24, 1963.

[2] Carlson, D. J. and Hoglund, R. F., "Particle Drag and Heat Transfer in Rocket Nozzles," *AIAA Journal,* Vol. 2, Nov. 1964, pp. 1980-1984.

[3] Crowe, C. T., "Drag Coefficient of Particles in a Rocket Nozzle," *AIAA Journal*, Vol. 5, May 1967, pp. 1021-1022.

[4] Crowe, C. T., Babcock, W. R., and Willoughby, P. G., "Drag Coefficient for Particles in Rarefied Low Mach Number Flows," International Symposium on Two-Phase Systems, Technion, Haifa, Israel, paper 3-3, 1971.

[5] Waldman, G. D., "Parametric Study for a Conceptual Multicomponent Flow System, TR 74-0030, May 1974, Aerospace Research Lab., Wright-Paterson Air Force Base, Ohio.

[6] Korkan, K. D., Petrie, S. L., and Bodonyi, R. J., "Particle Concentrations in High Mach Number Two-Phase Flows," AIAA Paper 74-606, Bethesda, Md., 1974

[7] Meyers, J. F. and Walsh, M. J., "Computer Simulation of a Fringe Type Laser Velocimeter," presented at the Workshop on Laser Velocimetry, Purdue University, Lafayette, Ind., March 27-29, 1974.

[8] Bailey, A. B. and Hiatt, J., "Free-Flight Measurements of Sphere Drag at Subsonic, Transonic, Supersonic, and Hypersonic Speeds for Continuum, Transition, and Near-Free-Molecular Flow Conditions," AEDC-TR-70-291, March 1971, Arnold Engineering Development Center, Tullahoma, Tenn.

[9] Zarin, N. A., "Measurement of Non-Continuum and Turbulence Effects on Subsonic Sphere Drag," NASA CR-1585, June 1970.

[10] Bailey, A. B., "Sphere Drag Coefficient for Subsonic Speeds in Continuum and Free-Molecular Flow," *Journal of Fluid Mechanics,* Vol. 65, Pt. 2, Aug. 1974, pp. 401-410

[11] Emmons, H. W., ed., *Fundamentals of Gas Dynamics, Vol. III. High Speed Aerodynamics and Jet Propulsion*, Princeton University Press, Princeton, N. J., 1958, p. 704.

[12] Torobin, L. B. and Gauvin, W. H., "Fundamental Aspects of Solids-Gas Flow, Part I; Introductory Concepts and Idealized Sphere Motion in Viscous Regime," *The Canadian Journal of Chemical Engineering,* Aug. 1959, pp. 129-141, Vol.37.